Effective cross-Kerr nonlinearity and robust phase gates with trapped ions

F. L. Semião[1, *] and A. Vidiella-Barranco[1, †]

[1]*Instituto de Física "Gleb Wataghin" - Universidade Estadual de Campinas, 13083-970 Campinas, São Paulo, Brazil*

(Dated: February 1, 2008)

We derive an effective Hamiltonian that describes a cross-Kerr type interaction in a system involving a two-level trapped ion coupled to the quantized field inside a cavity. We assume a large detuning between the ion and field (dispersive limit) and this results in an interaction Hamiltonian involving the product of the (bosonic) ionic vibrational motion and field number operators. We also demonstrate the feasibility of operation of a phase gate based on our hamiltonian. The gate is insensitive to spontaneous emission, an important feature for the practical implementation of quantum computing.

PACS numbers: 03.67.Lx, 42.65.Hw, 32.80.Qk

Nonlinear effects are specially relevant to quantum optics as well as to applications in quantum information and computation. In particular, the cross-Kerr coupling is at the basis of various schemes, such as quantum non-demolition measurements [1, 2], a Fock state synthesizer [3], entanglement purification [4], teleportation [5], quantum codewords [6], a proposal of realization of a CNOT gate [7], and Bell-state detection [8], for instance. Large cross-Kerr nonlinearities are usually required for the actual implementation of those schemes, but the values of the Kerr coefficients in the available nonlinear media are in general relatively small, although some progress based on atomic gases has been achieved [9, 10, 11]. It is therefore of interest to seek alternative ways of obtaining nonlinear interactions of that type. The cross-Kerr interaction may involve the effective coupling of two modes of the field [11], or even two sub-systems of different nature, such as an atom and a field [12].

Thinking about specific applications, and given the extraordinary development of the field of quantum computing, it is important to identify physical systems and situations that could be suitable for the realization of efficient quantum logic gates. A rather interesting physical system comprises a single trapped ion coupled to the quantized field [13]. That configuration forms a tripartite quantum system: the internal electronic degrees of freedom; the quantized center-of-mass motion; and the quantum cavity field. Several aspects of that system connected to quantum information and computation have been already investigated, e.g., a scheme for generation of entangled states involving the ionic vibrational motion and the quantized field [14], as well as an alternative proposition of implementation of a CNOT gate [15]. As a matter of fact, the possibility of coherent manipulation of trapped ions (qubits suitable for storing quantum information) coupled to photons (qubits suitable for carrying quantum information) [16] opens new possibilities for quantum information processing [17].

In this paper we derive, from a general hamiltonian involving the oscillating ion interacting with the quantized field, an effective Hamiltonian which couples the quantized field and the ionic motion in a way similar to what it is found in nonlinear optics. More specifically, we obtain a coupling which is basically the product of bosonic number operators referring to the vibrational motion ($\hat{a}^\dagger\hat{a}$) and the quantized field ($\hat{b}^\dagger\hat{b}$), yielding an interaction hamiltonian of the form $\hat{H}_{int} = \lambda\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}$. In the second part of our paper, we suggest an application of our Hamiltonian for quantum computing, presenting a proposal of implementation of a quantum phase gate. In our scheme, the ion is initially prepared in its ground (electronic) state $|g\rangle$, remaining in that state during the gate operation, which means that the gate will be quite robust against atomic spontaneous emission.

We consider a two level ion (with atomic frequency $\omega_{\rm a}$) trapped in a harmonic potential (frequency ν) inside a high finesse cavity, and coupled to a single mode (frequency $\omega_{\rm c}$) of the quantized cavity field. The ion has its motion well cooled down and therefore it should described quantum mechanically. The Hamiltonian for that system reads [13]

$$H = \nu\hat{a}^\dagger\hat{a} + \omega_{\rm c}\hat{b}^\dagger\hat{b} + \frac{1}{2}\omega_{\rm a}\hat{\sigma}_{\rm z} + g(\hat{\sigma}_+\hat{b} + \hat{b}^\dagger\hat{\sigma}_-)\cos\eta(\hat{a}^\dagger + \hat{a}), \tag{1}$$

where $\hat{a}^\dagger(\hat{a})$ are the creation(annihilation) operators relative to the excitations of the center-of-mass oscillatory motion, $\hat{b}^\dagger(\hat{b})$ are the creation (annihilation) operators of photons in the field mode, and g is the ion-field coupling constant. The operators $\hat{\sigma}_{\rm z}$, σ_+ ($\hat{\sigma}_-$) are the population difference and excitation (de-excitation) atomic operators, respectively. In order to derive a Kerr-like interaction we need to perform a couple of approximations. By one side, we note that the factor $\cos\eta(\hat{a}^\dagger + \hat{a})$ in Hamiltonian (1) may be expanded in powers of the operators $\hat{a}$ and $\hat{a}^\dagger$ giving then rise to different processes changing either energy or phase of the center-of-mass motion. It is then mandatory to get rid of the energy changing terms, keeping just terms proportional to powers of the number operator $\hat{a}^\dagger\hat{a}$. In the interaction picture, the Hamiltonian (1) is given by

$$\hat{H}_I = g(\hat{\sigma}_+\hat{b}e^{i\Delta t} + \hat{\sigma}_-\hat{b}^\dagger e^{-i\Delta t})\cos\eta(\hat{a}^\dagger e^{i\nu t} + \hat{a}e^{-i\nu t}), \tag{2}$$

where we defined $\Delta = \omega_{\rm a} - \omega_{\rm c}$. Making use of the relation $e^{\hat{A}+\hat{B}} = e^{-[\hat{A},\hat{B}]/2}e^{\hat{A}}e^{\hat{B}}$, valid for $[\hat{A},\hat{B}] = \gamma$, where γ is an

arbitrary complex number, we have

$$\hat{H}_I = g\hat{\sigma}_+\hat{b}\sum_{\alpha,\beta} F(\hat{a}^\dagger,\hat{a};\alpha,\beta)e^{i[\Delta+\nu(\alpha-\beta)]t} + g\hat{\sigma}_-\hat{b}^\dagger \sum_{\alpha,\beta} F(\hat{a}^\dagger,\hat{a};\alpha,\beta)e^{-i[\Delta+\nu(\alpha-\beta)]t}, \tag{3}$$

where we defined the function

$$F(\hat{a}^\dagger,\hat{a};\alpha,\beta) = \frac{e^{-\eta^2/2}}{2\,\alpha!\beta!}[(i\eta)^{\alpha+\beta} + (-i\eta)^{\alpha+\beta}]a^{\dagger\alpha}a^\beta. \tag{4}$$

Now, we perform the first approximation. By analyzing the temporal dependence in Hamiltonian (3), we may carefully choose the frequencies of the system in order to perform a suitable rotating wave approximation. In this approximation, rapidly oscillating terms are dropped out and just the slow ones are kept. The terms in (3) oscillate in time either as $e^{\pm i\Delta t}$ or $e^{i\pm(\Delta+k\nu)t}$, with k integer. In the regime $\Delta \ll \nu$ and $\Delta \neq k\nu$, the terms oscillating with lower frequencies are those proportional to $e^{\pm i\Delta t}$. This occurs if $\alpha = \beta$, or

$$F(\hat{a}^\dagger,\hat{a};\alpha,\alpha) = e^{-\eta^2/2}\frac{(-1)^\alpha\eta^{2\alpha}a^{\dagger\alpha}a^\alpha.}{\alpha!^2}, \tag{5}$$

and then

$$\hat{H} = \nu\hat{a}^\dagger\hat{a} + \omega_c\hat{b}^\dagger\hat{b} + \frac{\omega_a}{2}\hat{\sigma}_z + g(\hat{\sigma}_+\hat{b} + \hat{\sigma}_-\hat{b}^\dagger)f(\hat{a}^\dagger\hat{a}), \tag{6}$$

with

$$f(\hat{a}^\dagger\hat{a}) = e^{-\eta^2/2} : J_0(\mathfrak{n}) :, \tag{7}$$

where $: J_0(\mathfrak{n}) :$ is just the normally ordered zeroth order Bessel function of the first kind [18], with $\mathfrak{n} \equiv 2\eta\sqrt{\hat{a}^\dagger\hat{a}}$. We note the presence of $\hat{b}$ and $\hat{b}^\dagger$, which must be replaced by terms proportional to $\hat{b}^\dagger\hat{b}$. This may be accomplished by considering the large detuning limit $\Delta \gg g$. We proceed and write down the Heisenberg equations of motion for the atomic and field transition operators. From (1) one obtains

$$i\frac{d\hat{\sigma}_+}{dt} = -\omega_a\sigma_+ + g\hat{b}^\dagger\hat{\sigma}_z f(\hat{a}^\dagger\hat{a}) \tag{8}$$

$$i\frac{d\hat{b}}{dt} = \omega_c\hat{b} + g\hat{\sigma}_- f(\hat{a}^\dagger\hat{a}) \tag{9}$$

$$i\frac{d\hat{a}}{dt} = \nu\hat{a} + g(\hat{\sigma}_+\hat{b} + \hat{\sigma}_-\hat{b}^\dagger)\frac{\partial}{\partial\hat{a}^\dagger}f(\hat{a}^\dagger\hat{a}). \tag{10}$$

It is now suitable to move to a new frame using the transformation

$$\hat{\sigma}_+ \to \hat{\sigma}_+e^{i\omega_a t} \tag{11}$$

$$\hat{b}(t) \to \hat{b}e^{-i\omega_c t} \tag{12}$$

$$\hat{a}(t) \to \hat{a}e^{-i\nu t}, \tag{13}$$

so that the above equations of motion can be rewritten as

$$i\frac{d\hat{\sigma}_+}{dt} = g\hat{b}^\dagger\hat{\sigma}_z f(\hat{a}^\dagger\hat{a})e^{-i\Delta t} \tag{14}$$

$$i\frac{d\hat{b}}{dt} = g\hat{\sigma}_- f(\hat{a}^\dagger\hat{a})e^{-i\Delta t} \tag{15}$$

$$i\frac{d\hat{a}}{dt} = g(\hat{\sigma}_+\hat{b}e^{i\Delta t} + \hat{\sigma}_-\hat{b}^\dagger e^{-i\Delta t})e^{-i\nu t}\frac{\partial}{\partial\hat{a}^\dagger}f(\hat{a}^\dagger\hat{a}). \tag{16}$$

In the large detuning limit, i.e. for $\Delta = \omega_a - \omega_c \gg g$, we may perform the second order approximation which consists in assuming that the operators vary slowly when compared to the relevant frequencies involved. This allows us to integrate (14) obtaining

$$\hat{\sigma}_+ = \frac{ge^{-i\Delta t}}{\Delta}\hat{b}^\dagger\hat{\sigma}_z f(\hat{a}^\dagger\hat{a}). \tag{17}$$

After substituting (17) into (15), it results

$$i\frac{d\hat{b}}{dt} = \frac{g^2}{\Delta} f^2(\hat{a}^\dagger\hat{a})\hat{b}\hat{\sigma}_z. \tag{18}$$

This equation of motion can be obtained from the effective Hamiltonian

$$\hat{H}_{\text{eff}} = \nu\hat{a}^\dagger\hat{a} + \omega_c\hat{b}^\dagger\hat{b} + \frac{\omega_a}{2}\hat{\sigma}_z + \frac{g^2}{\Delta} f^2(\hat{a}^\dagger\hat{a})(\hat{\sigma}_+\hat{\sigma}_-\hat{b}\hat{b}^\dagger - \hat{\sigma}_-\hat{\sigma}_+\hat{b}^\dagger\hat{b}). \tag{19}$$

One could wonder now what is the validity of such effective Hamiltonian. Its limit of application may be found by using ordinary time-dependent perturbation theory. Once the effective Hamiltonian does not allow any transition between bare states we want the transitions $|g, n_c, m_{\text{v}}\rangle \to |e, (n-1)_c, m_{\text{v}}^{'}\rangle$ to be unlikely to happen. Mathematically, the transition probability is given by

$$P(t) = |\langle e, (n-1)_c, m_{\text{v}}^{'}|\hat{U}(t)|g, n_c, m_{\text{v}}\rangle|^2, \tag{20}$$

where $\hat{U}(t)$ is the time evolution operator. In first order,

$$\hat{U}(t) = \mathbb{1} - \imath\int_0^t \hat{H}(t^{'})\, dt^{'}, \tag{21}$$

being $\hat{H}(t)$ the original Hamiltonian (1) in the interaction picture. After a lengthy calculation, the wanted transition probability is found to be

$$P(t) = \left(\frac{g\, e^{-\eta^2/2}}{\Delta + \nu(m - m')}\right)^2 A(t) + \left(\frac{g\, e^{-\eta^2/2}}{\Delta - \nu(m - m')}\right)^2 B(t), \tag{22}$$

where $A(t)$ and $B(t)$ are oscillating functions of time. It is then clear that the effective Hamiltonian is to be safely applied when the conditions $\Delta \neq k\nu\ \ (k = 0, \pm 1, \pm 2, \ldots)$ and $g \ll \Delta$ are fulfilled, as well as $\Delta \ll \nu$ (for $m \neq m^{'}$).

The Hamiltonian in eq. (19) has novel and interesting features. It involves products of the number field operator $b^\dagger b$ with the vibrational number operator $a^\dagger a$, within the Bessel function J_0, characterizing a cross-Kerr type interaction between two bosons. The Stark shifts resulting from the large detuning condition we have assumed, gave rise to the interaction terms characteristic of the cross-Kerr interaction. The key point here is that the vibrational number operator $a^\dagger a$ comes out multiplying the light shifts. For a sufficiently small value of the Lamb-Dicke parameter η (Lamb-Dicke regime), we may truncate the Bessel function in order to obtain a term proportional to $\eta^2 a^\dagger a$. From (7) we note that

$$f^2(\hat{a}^\dagger\hat{a}) \approx 1 - \eta^2 - 2\eta^2\hat{a}^\dagger\hat{a}, \tag{23}$$

and then, if the ion is initially prepared (internal levels) in its ground state $|g\rangle$, we may rewrite (19) in that approximation as

$$\hat{H}_{\text{eff}}^{\text{LD}} = \nu\hat{a}^\dagger\hat{a} + \tilde{\omega}_c\hat{b}^\dagger\hat{b} + \lambda\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}, \tag{24}$$

where $\tilde{\omega}_c = \omega_c + \eta^2 g^2/\Delta - g^2/\Delta$ is a shift in the field frequency and the coupling constant between the bosonic modes is $\lambda = 2\eta^2 g^2/\Delta$. The interaction term of (24) is precisely the sought Kerr-type interaction in its well known form. By careful choice of the η, g and Δ it is possible to obtain a considerable non-linearity. According to [19], a realistic choice for the ion-field coupling could be $g = 2\pi \times 1.51$MHz. In the Lamb-Dicke regime, we may consider $\eta = 0.05$ and for the high detuning limit $\Delta = 10g$ is also adequate. These values lead to $\lambda \approx 5$ kHz.

Now, we would like to show the phase gate operation using our effective hamiltonian. We would like to point out that other few proposals may be found in the literature, involving trapped ions [15, 20], cavity QED [21] or optical lattices [22]. There are also some experimental realizations [23]. Here the quantum subsystems involved are the ionic center-of-mass (phonons) and the quantized field (photons). Considering that the ion is in its electronic groud state, the Hamiltonian (19) assumes the following form in the interaction representation

$$H_{\text{eff}} = -\frac{g^2}{\Delta} f^2(\hat{a}^\dagger\hat{a})\hat{b}^\dagger\hat{b}, \tag{25}$$

which is adequate for producing general phase shifts. Because $[a^\dagger a, H_{\text{eff}}] = [b^\dagger b, H_{\text{eff}}] = 0$, we may write the evolution operator $U(t) = \exp(-iH_{\text{eff}}t/\hbar)$ in diagonal form in the basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$, with $|ij\rangle \equiv |\text{ion}\rangle \otimes |\text{field}\rangle$. Therefore

$$\begin{aligned}\hat{U}(t) &= \exp\left\{it\frac{g^2 e^{-\eta^2}}{\Delta}[:J_0(\mathfrak{n}):]^2\hat{b}^\dagger\hat{b}\right\} \\ &= e^{i\phi_{00}}|00\rangle\langle 00| + e^{i\phi_{10}}|10\rangle\langle 10| + e^{i\phi_{01}}|01\rangle\langle 01| + e^{i\phi_{11}}|11\rangle\langle 11|,\end{aligned} \tag{26}$$

where

$$\begin{aligned}\phi_{00} &= 0\\ \phi_{10} &= 0\\ \phi_{01} &= \Omega t\\ \phi_{11} &= \Omega t[L_1(\eta^2)]^2,\end{aligned} \quad (27)$$

being $L_1(\eta^2)$ the first order Laguerre polynomial and $\Omega = -g^2 e^{-\eta^2}/\Delta$. In order to have a π phase gate we need to have $\Omega t = 2\pi$ and $\phi_{01} - \phi_{11} = -\pi$. This would demand $[L_1(\eta^2)]^2 = 1/2$, or $\eta \approx 0.54$, which is close to realistic values for the Lamb-Dicke parameter found in the literature [24].

It is now necessary to give a brief discussion on the experimental implementation of the gate. According to recent experimental data [19], the transition from the state $P_{3/2}$ to $D_{5/2}$ in ^{40}Ca^{+} ions is coupled to the vacuum field with strength $g = 2\pi \times 1.51$ MHz. The losses are mainly due to the cavity at rate $\kappa = 2\pi \times 41.7$ KHz and atomic spontaneous emission at rate $\gamma = 2\pi \times 1.58$ MHz. Choosing $\Delta = 10g$, the gate operation time is $t_{op} \approx 9\,\mu$s. We must compare it to the decay rates. Firstly, quite remarkable is the fact that the gate works all time with the ion in the ground state. For this reason the proposal is quite insensitive to spontaneous emission and this is much desirable. Secondly, when comparing it to $\kappa^{-1} = 3.8\,\mu$s, one can see that they are at the same order and so that it is still necessary to improve the quality of the actual cavities. However, rather than improving the cavity finesse, one could think of using atomic levels which would provide a stronger coupling g. The important point to be noticed here is that the robustness against spontaneous emission in our scheme is the key to improve the coupling constant g. We remember that this coupling is dependent on the atomic decay rate as $g \backsim \sqrt{\gamma}$ and then one could choose fast decaying excited levels leading to stronger couplings and consequently faster gates. It is worthwhile to notice that a phase gate in the Lamb-Dicke regime [see Eq.(24)] would be much faster than the one we have just discussed. In that case, $t_{op} \approx 0.62$ms which means that it is possible to have about 160 cycles per κ^-1, making our proposals even more interesting for practical implementation. Having taken those typical experimental values into account, and considering the robustness against spontaneous emission, we find that our proposal opens up new possibilities for feasible quantum computing and also brings the interesting possibility of implementing cross-Kerr interactions between bosons in a well developed experimental system.

The authors would like to thank P.P. Munhoz for a critical reading of the manuscript. This work is partially supported by CNPq (Conselho Nacional para o Desenvolvimento Científico e Tecnológico), and FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo) grant number 02/02715-2, Brazil.

* Electronic address: semiao@ifi.unicamp.br
† Electronic address: vidiella@ifi.unicamp.br

[1] N. Imoto, H.A. Haus, and Y. Yamamoto, Phys. Rev. A **32**, 2287 (1985).
[2] P. Grangier, J.A. Levenson, and J.P. Poizat, Nature (London) **396**, 537 (1998).
[3] G.M. D'Ariano, L. Maccone, M.G.A. Paris, and M.F. Sacchi, Phys. Rev. A **61**, 053817 (2000).
[4] L.M. Duan, G. Giedke, J.I Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 4002 (2000).
[5] D. Vitali, M. Fortunato, and P. Tombesi, Phys. Rev. Lett. **85**, 445 (2000).
[6] S. Pirandola, S. Mancini, D. Vitali and P. Tombesi, Europhys. Lett. **68**, 323 (2004).
[7] K. Nemoto, and W.J. Munro, Phys. Rev. Lett. **93**, 250502 (2004).
[8] S. D. Barrett, Pieter Kok, Kae Nemoto, R. G. Beausoleil, W. J. Munro, and T. P. Spiller, Phys. Rev. A **71**, 060302 (2005)
[9] L. V. Hau, S. E. Harris, Z. Dutton, and C. H. Behroozi, Nature (London) **397**, 594 (1999).
[10] A. Imamoglu, H. Schmidt, G. Woods, and M. Deutsch, Phys. Rev. Lett. **79**, 1467 (1999).
[11] C. Ottaviani, D. Vitali, M. Artoni, F. Cataliotti, and P. Tombesi, Phys. Rev. Lett. **90**, 197902 (2003).
[12] J. Zhang, K. Peng, and S.L. Braunstein, Phys. Rev. A **68**, 035802 (2003).
[13] H. Zeng and F. Lin, Phys. Rev. A **50**, R3589 (1994); V. Bužek, G. Drobný, M.S. Kim, G. Adam, and P.L. Knight, Phys. Rev. A **56**, 2352 (1998).
[14] F.L. Semião, A. Vidiella-Barranco and J.A. Roversi, Phys. Rev. A **64**, 024305 (2001).
[15] F.L. Semião, A. Vidiella-Barranco and J.A. Roversi, Phys. Lett. A, **299**, 423 (2002).
[16] A.B. Mundt, A. Kreuter, C. Becher, D. Leibfried, J. Eschner, F. Schmidt-Kaler, and R. Blatt, Phys. Rev. Lett. **89**, 103001 (2002).
[17] A.B. Mundt, A. Kreuter, C. Russo, C. Becher, D. Leibfried, J. Eschner, F. Schmidt-Kaler, and R. Blatt, App. Phys. B **76**, 117 (2003).
[18] K. Banaszek and K. Wódkiewicz, Phys. Rev. A **55**, 3117 (1997).
[19] C. Maurer, C. Becher, C. Russo, J. Eschner and R. Blatt, New J. Phys. 6, 94 (2004).
[20] E. Solano, M. França Santos, and P. Milman, Phys. Rev. A **64**, 024304 (2001).
[21] M.S. Zubairy, M. Kim, and M.O. Scully, Phys. Rev. A **68**, 033820 (2003).
[22] J. K. Pachos and P.L. Knight, Phys. Rev. Lett. **91**, 107902 (2003).

[23] Q. A. Turchette, C. J. Hood, W. Lange, H. Mabuchi, and H. J. Kimble, Phys. Rev. Lett. **75**, 4710 (1995); A. Rauschenbeutel, G. Nogues, S. Osnaghi, P. Bertet, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **83**, 5166 (1999); D. Leibfried, B. DeMarco, V. Meyer, D. Lucas, M. Barrett, J. Britton, W. M. Itano, B. Jelenkovic, C. Langer, T. Rosenband, D. J. Wineland, Nature (London) **422**, 412 (2003).

[24] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, Rev. Mod. Phys. **75**, 281 (2003).